

THURSTON'S BOUNDARY FOR TEICHMÜLLER SPACES OF INFINITE SURFACES: THE LENGTH SPECTRUM

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ABSTRACT. Let X be an infinite geodesically complete hyperbolic surface which can be decomposed into geodesic pairs of pants. We introduce Thurston's boundary to the Teichmüller space $T(X)$ of the surface X using the length spectrum analogous to Thurston's construction for finite surfaces. Thurston's boundary using the length spectrum of X is a "closure" of projective bounded measured laminations $PML_{bdd}(X)$, and it coincides with $PML_{bdd}(X)$ when X can be decomposed into a countable union of geodesic pairs of pants whose boundary geodesics $\{\alpha_n\}_{n \in \mathbb{N}}$ have lengths pinched between two positive constants. When a subsequence of the lengths of the boundary curves of the geodesic pairs of pants $\{\alpha_n\}_n$ converges to zero, Thurston's boundary using the length spectrum is strictly larger than $PML_{bdd}(X)$.

1. INTRODUCTION

Fix a geodesically complete infinite area hyperbolic surface X_0 . The space of all quasiconformal deformations of X modulo conformal maps and homotopies is an infinite-dimensional manifold called the Teichmüller space $T(X_0)$ of X_0 . We study the limiting behaviour of the quasiconformal deformations of X when the dilations of the quasiconformal maps increase without bound using the length spectrum. Thurston [25], [10] used the length spectrum to compactify the Teichmüller space of a closed surface by adding to it the space of projective measured laminations of the surface. Bonahon [6] used geodesic currents to give an alternative description of Thurston's boundary for the Teichmüller space of a closed surface. In [23], geodesic currents were used to introduce Thurston's boundary to $T(X)$.

A complete hyperbolic surface is obtained by gluing geodesic pairs of pants (with possible at most two punctures on the boundary) and by adding at most countably many funnels with closed geodesic boundary and half-planes with boundary infinite geodesics (cf. [4]). Since we consider the length spectrum, it is natural to restrict our attention to geodesically complete, infinite hyperbolic surfaces that are obtained by gluing countably many geodesic pairs of pants (cf. [24], [3], [1]). For such surfaces, the Teichmüller space is completely determined by the marked length spectrum. The study of the length spectrum properties for infinite surfaces is started by Shiga [24], and it was further developed by various authors (e.g. [1], [2], [3] [15], [13], [18], ...).

Denote by $[f] \in T(X_0)$ the equivalence class of a quasiconformal map $f : X_0 \rightarrow X$. Let \mathcal{S} be the set of all simple closed geodesics on X_0 . Homotopy class of a quasiconformal map $f : X_0 \rightarrow X$ induces a function from \mathcal{S} to \mathbb{R} which assigns to

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each $\alpha \in \mathcal{S}$ the length of a geodesic in X that is homotopic to $f(\alpha)$. Thus we have an injective map

$$\mathcal{X} : T(X_0) \rightarrow \mathbb{R}_{\geq 0}^{\mathcal{S}}.$$

When X_0 is a closed hyperbolic surface then the above map is a homeomorphism onto its image if $\mathbb{R}_{\geq 0}^{\mathcal{S}}$ is equipped with the weak* topology (cf. [10]). In the case of an infinite surface with a geodesic pants decomposition, the *length spectrum metric* is defined by (cf. [24], [2])

$$d_{ls}([f], [g]) = \sup_{\alpha \in \mathcal{S}} \left| \log \frac{l_{f(X_0)}(f(\alpha))}{l_{g(X_0)}(g(\alpha))} \right|.$$

Shiga [24] proved that the topology induced by the length spectrum metric on $T(X_0)$ is equal to the Teichmüller topology when the surface X has a geodesic pants decomposition with lengths of boundary geodesics (of the pants) pinched between two positive constants. Allessandrini, Liu, Papadopoulos and Su [1] proved that the length spectrum on $T(X_0)$ is not complete when X_0 contains a sequence of simple closed geodesics whose length goes to zero. Thus the two topologies in this case are different.

We introduce a *normalized supremum norm* on $\mathbb{R}_{\geq 0}^{\mathcal{S}}$ by

$$\|f\|_{\infty}^{\text{norm}} = \sup_{\alpha \in \mathcal{S}} \left| \frac{f(\alpha)}{l_{X_0}(\alpha)} \right|$$

for $f \in \mathbb{R}_{\geq 0}^{\mathcal{S}}$. The normalized supremum norm on $\mathbb{R}_{\geq 0}^{\mathcal{S}}$ makes the map $\mathcal{X} : T(X_0) \rightarrow \mathbb{R}_{\geq 0}^{\mathcal{S}}$ a homeomorphism onto its image (cf. Lemma 4.1).

Analogous to the closed surface case, we projectivize \mathcal{X} and obtain an injective map

$$P\mathcal{X} : T(X_0) \rightarrow P\mathbb{R}_{\geq 0}^{\mathcal{S}}.$$

By definition, *(length spectrum) Thurston's boundary* of $T(X_0)$ consists of the boundary points of the image $P\mathcal{X}(T(X_0))$ of $T(X_0)$, where $P\mathbb{R}_{\geq 0}^{\mathcal{S}}$ is given the quotient topology with respect to the normalized supremum norm on $\mathbb{R}_{\geq 0}^{\mathcal{S}}$.

Theorem 1. *Let X_0 be an infinite area geodesically complete hyperbolic surface that has a geodesic pants decomposition with boundary geodesics of pair of pants $\{\alpha_n\}_{n \in \mathbb{N}}$. Then (length spectrum) Thurston's boundary of $T(X_0)$ is the closure of the space of projective bounded measured laminations $PML_{bdd}(X_0)$ in $P\mathbb{R}^{\mathcal{S}}$, where $P\mathbb{R}^{\mathcal{S}}$ has the quotient topology induced by the topology on $\mathbb{R}^{\mathcal{S}}$ coming from the normalized supremum norm.*

If the lengths of $\{\alpha_n\}_{n \in \mathbb{N}}$ are pinched between two positive constants then length spectrum Thurston's boundary is equal to $PML_{bdd}(X_0)$.

If the lengths of $\{\alpha_n\}_{n \in \mathbb{N}}$ are bounded from the above and there exists a subsequence $\{\alpha_{n_k}\}$ whose lengths converge to 0, then length spectrum Thurston's boundary is strictly larger than $PML_{bdd}(X_0)$.

In addition, Thurston's boundary of a hyperbolic surface X_0 whose every geodesic pants decomposition does not have an upper bound on the lengths of cuffs but that can be decomposed into bounded polygons with at most n sides (introduced by Kinjo [13]) equals $PML_{bdd}(X_0)$. On the other hand, if X_0 is the surface constructed by Shiga [24] such that the length spectrum metric is incomplete, then length spectrum Thurston's boundary is strictly larger than $PML_{bdd}(X_0)$ (cf. §4.4).

Recall that the quasiconformal Mapping Class Group $MCG_{qc}(X_0)$ consists of all quasiconformal maps $g : X_0 \rightarrow X_0$ up to homotopy (cf. [11]). The action of $MCG_{qc}(X_0)$ on the Teichmüller space $T(X_0)$ is given by $[f] \mapsto [f \circ g^{-1}]$ and it is continuous in the Teichmüller metric. Therefore it is also continuous in the length spectrum metric. The normalised supremum norm on $\mathbb{R}_{\geq 0}^{\mathcal{S}}$ is invariant under the change of markings of \mathcal{S} because it is the supremum over all simple closed curves \mathcal{S} . Therefore we obtain

Theorem 2. *The action of the quasiconformal Mapping Class Group $MCG_{qc}(X_0)$ on the Teichmüller space $T(X_0)$ extends to a continuous action on (length spectrum) Thurston's closure of $T(X_0)$.*

2. TEICHMÜLLER SPACES OF GEOMETRICALLY INFINITE HYPERBOLIC SURFACES

Let X_0 be a geodesically complete hyperbolic surface whose area is infinite. The universal covering \tilde{X}_0 of the surface X_0 is isometrically identified with the hyperbolic plane \mathbb{H} . The boundary at infinity $\partial_\infty \tilde{X}_0$ is identified with the unit circle S^1 .

The *Teichmüller space* $T(X_0)$ of the surface X_0 is the space of equivalence classes of all quasiconformal maps $f : X_0 \rightarrow X$ where X is an arbitrary complete hyperbolic surface modulo an equivalence relation. Two quasiconformal maps $f_1 : X_0 \rightarrow X_1$ and $f_2 : X_0 \rightarrow X_2$ are *equivalent* if there exists an isometry $I : X_1 \rightarrow X_2$ such that $f_2^{-1} \circ I \circ f_1$ is homotopic to the identity under a bounded homotopy. Denote by $[f]$ the equivalence class of a quasiconformal map $f : X_0 \rightarrow X$.

The *Teichmüller distance* on $T(X_0)$ is defined by

$$d_T([f_1], [f_2]) = \frac{1}{2} \log \inf_{g \simeq f_2 \circ f_1^{-1}} K(g)$$

where the infimum is taken over all quasiconformal maps g homotopic to $f_2 \circ f_1^{-1}$ and $K(g)$ is the quasiconformal constant of g . The *Teichmüller topology* on $T(X_0)$ is the topology induced by the Teichmüller distance.

3. MEASURED LAMINATIONS AND EARTHQUAKES

A geodesic lamination on a hyperbolic surface X is a closed subset of X that is foliated by non-intersecting complete geodesics called *leaves* of the lamination. Geodesic lamination on X lifts to a geodesic lamination on \mathbb{H} that is invariant under the action of the covering group of X . A *stratum* of a geodesic lamination is either a leaf of the lamination or a connected component of the complement. A connected component of the complement of a geodesic lamination in \mathbb{H} is isometric to a possibly infinite sided geodesic polygon whose sides are complete geodesics and possibly arcs on S^1 .

A *measured lamination* μ on X is an assignment of a positive Borel measure on each arc transverse to a geodesic lamination $|\mu|$ that is invariant under homotopies relative leaves of $|\mu|$. The geodesic lamination $|\mu|$ is called the *support* of μ . A measured lamination on X lifts to a measured lamination on \mathbb{H} that is invariant under the covering group of X .

A *left earthquake* $E : X_0 \rightarrow X$ with support geodesic lamination λ is a surjective map that is isometry on each stratum of λ such that each stratum is moved to the left relative to any other stratum. An earthquake of X_0 lifts to an earthquake of \mathbb{H} where the support is the lift of the support on X_0 (cf. Thurston [25]).

We give a definition of a (left) earthquake $E : \mathbb{H} \rightarrow \mathbb{H}$ with support geodesic lamination λ on \mathbb{H} . A *left earthquake* $E : \mathbb{H} \rightarrow \mathbb{H}$ is a bijection of \mathbb{H} whose restriction to any stratum of λ is an isometry of \mathbb{H} ; if A and B are two strata of λ then

$$(E|_A)^{-1} \circ E|_B$$

is a hyperbolic translation whose axis weakly separates A and B that moves B to the left as seen from A (cf. Thurston [25]).

An earthquake $E : \mathbb{H} \rightarrow \mathbb{H}$ induces a transverse measure μ to its support λ which defines a measured lamination μ with $|\mu| = \lambda$ (cf. [25]). An earthquake of \mathbb{H} extends by continuity to a homeomorphism of S^1 . Thurston's earthquake theorem states that any homeomorphism of S^1 can be obtained by continuous extension of a left earthquake (cf. Thurston [25]).

Given a measured lamination μ , there exists a map $E^\mu : \mathbb{H} \rightarrow \mathbb{H}$ whose transverse measure is μ and that satisfies all properties in the definition of an earthquake of \mathbb{H} except being onto (cf. [25], [12]). E^μ is uniquely determined by μ up to post-composition by an isometry of \mathbb{H} .

We define *Thurston's norm* of a measured lamination μ as

$$\|\mu\|_{Th} = \sup_J i(\mu, J)$$

where the supremum is over all hyperbolic arcs J of length 1.

A quasiconformal map of X lifts to a quasiconformal map of \mathbb{H} and the later extends to a quasisymmetric map of S^1 . Therefore we consider measured laminations whose earthquakes induces quasisymmetric maps of S^1 . An earthquake E^μ extends by continuity to a quasisymmetric map of S^1 if and only if $\|\mu\|_{Th} < \infty$ (cf. [25], [12], [18], [19]).

Denote by $ML_{bdd}(X)$ and $ML_{bdd}(\mathbb{H})$ the space of all measured laminations with finite Thurston's norm on X and \mathbb{H} , respectively. The above result gives a bijective map

$$EM : T(X) \rightarrow ML_{bdd}(X).$$

Note that $\|t\mu\|_{Th} = t\|\mu\|_{Th}$, for $t > 0$. Then, for $\|\mu\|_{Th} < \infty$, we have that $t \mapsto E^{t\mu}$, for $t > 0$, is a path in $T(X)$ called *earthquake path*.

4. THURSTON'S BOUNDARY FOR TEICHMÜLLER SPACES OF INFINITE SURFACES USING THE LENGTH SPECTRUM

In this section we consider infinite type hyperbolic surfaces and introduce "length spectrum" Thurston's boundary to their Teichmüller spaces. It turns out that length spectrum Thurston's boundary differs from Thurston's boundary introduced using geodesic currents (see [23] for the construction using geodesic currents).

Recall that X_0 is a geodesically complete hyperbolic surface that has a geodesic pants decomposition. In other words, X_0 is formed by gluing infinitely many geodesic pairs of pants along their boundaries.

Let $\{\alpha_n\}_{n \in \mathbb{N}}$ be the family of cuffs (i.e. boundary components) of a geodesic pants decomposition of X_0 as above. Then each α_n is a simple closed geodesic or a puncture. We say that $\{\alpha_n\}$ is an *upper-bounded geodesic pants decomposition* of X_0 if there exists $M > 0$ such that, for each $n \in \mathbb{N}$,

$$l_{X_0}(\alpha_n) \leq M$$

where $l_{X_0}(\alpha_n)$ is the length of α_n for the hyperbolic metric of X_0 (cf. [1]). Moreover, $\{\alpha_n\}$ is a *lower-bounded geodesic pants decomposition* of X_0 if there exists $m > 0$ such that, for each $n \in \mathbb{N}$,

$$l_{X_0}(\alpha_n) \geq m.$$

4.1. General infinite surfaces. Denote by \mathcal{S} the set of all simple closed geodesics on a geodesically complete hyperbolic surface X_0 equipped with a geodesic pants decomposition. Let $\mathbb{R}_{\geq 0}^{\mathcal{S}}$ be the space of non-negative functions on the set of all simple closed geodesics \mathcal{S} of X_0 . We define a map \mathcal{X} from the Teichmüller space $T(X_0)$ into $\mathbb{R}_{\geq 0}^{\mathcal{S}}$, for $[f] \in T(X_0)$ and $\alpha \in \mathcal{S}$,

$$\mathcal{X}([f])(\alpha) = l_{f(X_0)}(f(\alpha)),$$

where $f(X_0)$ is the image hyperbolic surface under quasiconformal mapping f and $l_{f(X_0)}(f(\alpha))$ is the length of a simple closed geodesic on $f(X_0)$ homotopic to a simple closed curve $f(\alpha)$. The map $\mathcal{X} : T(X_0) \rightarrow \mathbb{R}_{\geq 0}^{\mathcal{S}}$ is injective.

The *length spectrum metric* on $T(X_0)$ is given by

$$d_{ls}([f_1], [f_2]) = \sup_{\delta \in \mathcal{S}} \left\{ \left| \log \frac{l_{f_2(X_0)}(f_2(\delta))}{l_{f_1(X_0)}(f_1(\delta))} \right| \right\}.$$

Shiga [24] proved that if X_0 has an upper and lower bounded geodesic pants decomposition $\{\alpha_n\}_{n \in \mathbb{N}}$ then the Teichmüller distance induces the same topology as the length-spectrum distance on $T(X_0)$.

We introduce the *normalized supremum norm* on $\mathbb{R}^{\mathcal{S}}$ by

$$\|f\|_{\infty}^{norm} = \sup_{\delta \in \mathcal{S}} \frac{|f(\delta)|}{l_{X_0}(\delta)}$$

for all $f \in \mathbb{R}^{\mathcal{S}}$. Note that the normalized supremum norm on $\mathbb{R}^{\mathcal{S}}$ is infinite at some points of $\mathbb{R}^{\mathcal{S}}$. We consider only the subset of $\mathbb{R}^{\mathcal{S}}$ where the normalized supremum is finite and, for simplicity, denote it by $\mathbb{R}^{\mathcal{S}}$.

Proposition 4.1. *The length spectrum metric on $T(X_0)$ is locally bi-Lipschitz equivalent to the normalized supremum norm on $\mathcal{X}(T(X_0))$.*

Proof. Indeed, if

$$\sup_{\delta \in \mathcal{S}} \left| \frac{l_{f_1(X_0)}(f_1(\delta))}{l_{X_0}(\delta)} - \frac{l_{f_2(X_0)}(f_2(\delta))}{l_{X_0}(\delta)} \right| < \epsilon$$

then

$$\sup_{\delta \in \mathcal{S}} \left| \frac{l_{f_1(X_0)}(f_1(\delta))}{l_{X_0}(\delta)} \left| 1 - \frac{l_{f_2(X_0)}(f_2(\delta))}{l_{f_1(X_0)}(f_1(\delta))} \right| \right| < \epsilon.$$

Since f_1 is a quasiconformal map, there exists $M > 1$ such that $1/M \leq \frac{l_{f_1(X_0)}(f_1(\delta))}{l_{X_0}(\delta)} \leq M$ (cf. Wolpert [27]). The above and symmetry implies

$$\left| \frac{l_{f_2(X_0)}(f_2(\delta))}{l_{f_1(X_0)}(f_1(\delta))} - 1 \right|, \left| \frac{l_{f_1(X_0)}(f_1(\delta))}{l_{f_2(X_0)}(f_2(\delta))} - 1 \right| \leq M\epsilon$$

for all $\delta \in \mathcal{S}$, and one direction is obtained since $|\log x|/|x - 1|$ is between two positive constants for $1/2 < x < 2$. The other direction is obtained by reversing the order of the above inequalities and the two metrics are locally bi-Lipschitz. \square

Allessandrini, Liu, Papadopoulos and Su [2] proved that $T(X_0)$ is not complete in the length spectrum metric when there exists a sequence of simple closed geodesics on X_0 whose lengths converge to 0. Thus, $\mathcal{X} : T(X_0) \rightarrow \mathbb{R}_{\geq 0}^S$ is not a homeomorphism onto its image for the normalized supremum norm on $\mathbb{R}_{\geq 0}^S$ and the Teichmüller metric on $T(X_0)$ when X_0 contains a sequence of simple closed geodesics whose lengths converge to zero.

Denote by

$$\mathcal{PX} : T(X_0) \rightarrow P\mathbb{R}_{\geq 0}^S$$

the map from $T(X_0)$ into the projective space $P\mathbb{R}_{\geq 0}^S = (\mathbb{R}_{\geq 0}^S - \{\bar{0}\})/\mathbb{R}_{>0}$. The map \mathcal{PX} is injective on $T(X_0)$. The *length spectrum Thurston's boundary* of $T(X_0)$ is, by the definition, the space of all limit points in $P\mathbb{R}_{\geq 0}^S$ of the set $\mathcal{PX}(T(X_0))$ for the topology induced by the normalized supremum norm (c.f. [10] for the original Thurston's discussion on closed surfaces).

Note that a measured lamination μ on X_0 represent an element in $\mathbb{R}_{\geq 0}^S$ by the formula

$$\mu(\alpha) = i(\mu, \alpha)$$

for all $\alpha \in \mathcal{S}$, where $i(\mu, \alpha)$ is the intersection number.

Proposition 4.2. *Let X_0 be a geodesically complete infinite hyperbolic surface equipped with a geodesic pants decomposition. Then length spectrum Thurston's boundary of $T(X_0)$ contains the space of projective bounded measured lamination $PML_{bdd}(X_0)$ and it equals the closure of $PML_{bdd}(X_0)$ for the topology on $P\mathbb{R}_{\geq 0}^S$ induced by the normalized supremum norm.*

Proof. Let $\mu \in ML_{bdd}(X_0)$ be a bounded measured lamination on X_0 . Denote by $E^{t\mu}$, for $t > 0$, an earthquake path with the earthquake measure $t\mu$. Then $t \mapsto E^{t\mu}(X_0)$ is an analytic path in $T(X_0)$ because $\mu \in ML_{bdd}(X_0)$ (cf. [18]). Let f_t be a quasiconformal map from X_0 to X_t which is homotopic to $E^{t\mu}$.

For $\alpha \in \mathcal{S}$, the inequality

$$l_{f_t(X_0)}(f_t(\alpha)) \leq ti(\mu, \alpha) + l_{X_0}(\alpha)$$

implies that

$$(1) \quad \frac{\frac{1}{t}\mathcal{X}([f_t])(\alpha) - i(\mu, \alpha)}{l_{X_0}(\alpha)} \leq \frac{1}{t}$$

for all $\alpha \in \mathcal{S}$ and all $t > 0$.

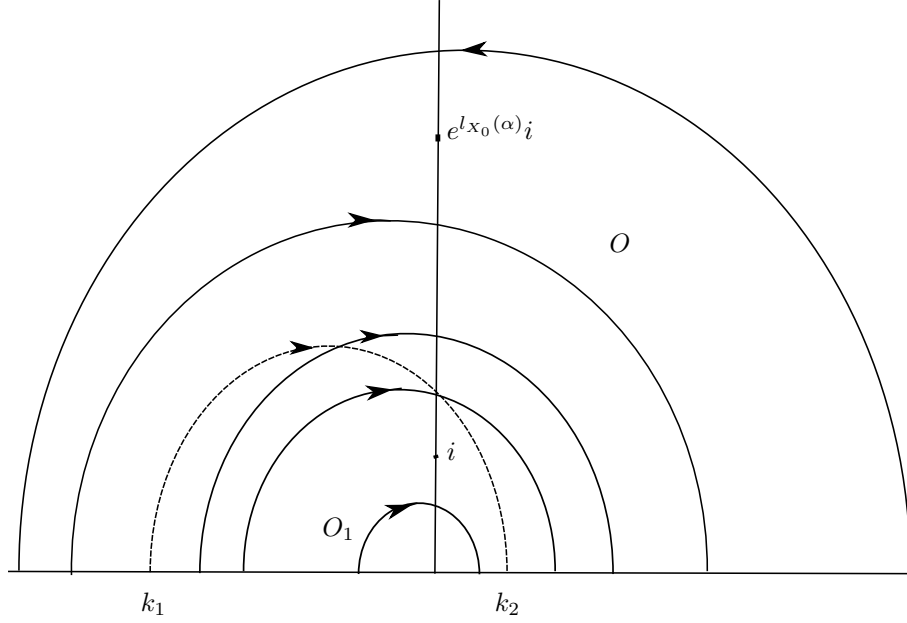
To obtain the opposite inequality, we choose the universal covering of X_0 such that $B(z) = e^{-l_{X_0}(\alpha)}z$ is a cover transformation corresponding to α . Let O be the stratum of the lift $\tilde{\mu}$ of μ to the universal covering \mathbb{H} that contains $e^{l_{X_0}(\alpha)}i$, and let O_1 be the stratum of $\tilde{\mu}$ that contains i . Normalize the earthquake $E^{t\mu}$ such that $E^{t\mu}|_O = id$. Then

$$B^t = E^{t\mu}|_{O_1} \circ B$$

is a covering transformation that corresponds to the geodesic on $f_t(X_0)$ homotopic to $f_t(\alpha)$ (cf. [8]). Denote by l_t the translation length of B_t and $l = l_{X_0}(\alpha)$ the translation length of B . Let $k_1 < 0$ and $k_2 > 0$ be the endpoints of the hyperbolic translation $E^{t\mu}|_{O_1}$, and let m_t be its translation length (cf. Figure 1).

A direct computation (cf. [22]) gives

$$trace(B^t) = 2 \cosh \frac{m_t - l}{2} - \frac{2k_1}{k_2 - k_1} \left(\cosh \frac{m_t + l}{2} - \cosh \frac{m_t - l}{2} \right)$$

FIGURE 1. Computing $E^{t\tilde{\mu}}|_{O_1}$

Consequently

$$2 \cosh \frac{l_t}{2} = \text{trace}(B^t) \geq 2 \cosh \frac{m_t - l}{2}$$

which implies

$$l_t \geq m_t - l.$$

Since the translation length of a composition of two hyperbolic translations (with non-intersecting axis and translating in the same direction) is at least as large as the sum of their translation lengths (cf. [25]), it follows that

$$m_t \geq ti(\mu, \alpha).$$

The above two inequalities give

$$\frac{1}{t} \frac{l_t}{l} \geq \frac{i(\mu, \alpha)}{l} - \frac{1}{t}$$

which implies

$$(2) \quad \frac{1}{t} \frac{\mathcal{X}([f_t])(\alpha)}{l_{X_0}(\alpha)} - \frac{i(\mu, \alpha)}{l_{X_0}(\alpha)} \geq -\frac{1}{t}.$$

Then equations (1) and (2) give that, uniformly in $\alpha \in \mathcal{S}$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \frac{\mathcal{X}([f_t])(\alpha)}{l_{X_0}(\alpha)} = \frac{i(\mu, \alpha)}{l_{X_0}(\alpha)}.$$

We established that each point in $PML_{bdd}(X_0)$ is in Thurston's boundary.

Let $\sigma \in \mathbb{R}_{\geq 0}^{\mathcal{S}}$ be such that its projective class $[\sigma]$ is in length spectrum Thurston's boundary. We need to establish that $[\sigma]$ is in the closure of $PML_{bdd}(X_0)$ for the normalized supremum norm.

There exists a sequence $[f_n] \in T(X_0)$ that converges to the projective class $[\sigma] \in P\mathbb{R}_{\geq 0}^S$. Let $t_n \rightarrow \infty$ as $n \rightarrow \infty$ be such that $\frac{1}{t_n}\mathcal{X}([f_n]) \rightarrow \sigma$ as $n \rightarrow \infty$ in the normalized supremum norm. Necessarily we have $\sup_n \|\frac{1}{t_n}\mathcal{X}([f_n])\|_{\infty}^{norm} < \infty$.

Let f_n be represented by a sequence of earthquakes $E^{t'_n \mu_n}$ with $\|\mu_n\|_{Th} = 1$ and $t'_n > 0$. Then $t'_n \rightarrow \infty$ as $n \rightarrow \infty$ and the first part of the proof gives

$$\|\frac{1}{t'_n}\mathcal{X}([f_n]) - \mu_n\|_{\infty}^{norm} < \frac{1}{t'_n}.$$

Note that if $\|\mu_n\|_{Th} = 1$ then $\|\mu_n\|_{\infty}^{norm} \leq 2$. Then the above inequality implies that $\|\frac{1}{t'_n}\mathcal{X}([f_n])\|_{\infty}^{norm} \leq 3$ for all t'_n with n large enough and the sequence $\frac{t'_n}{t_n}$ is bounded from the above and below by positive numbers. By choosing a subsequence, if necessary, we can assume that $\frac{t'_n}{t_n} \rightarrow c > 0$ as $n \rightarrow \infty$. It follows that, as $n \rightarrow \infty$,

$$\|\frac{1}{t_n}\mathcal{X}([f_n]) - c\mu_n\|_{\infty}^{norm} \rightarrow 0$$

which implies

$$\|c\mu_n - \sigma\|_{\infty}^{norm} \rightarrow 0$$

and the proof is completed. \square

4.2. Infinite surfaces with bounded geodesic pants decompositions. We consider a hyperbolic surface X_0 which can be decomposed into geodesic pairs of pants with cuffs $\{\alpha_n\}_{n \in \mathbb{N}}$ such that

$$1/M \leq l_{X_0}(\alpha_n) \leq M$$

for some $M > 1$ and for all $n \in \mathbb{N}$. We say that such X_0 has a *bounded geodesic pants decomposition*. The next proposition establishes that length spectrum Thurston's boundary coincides with Thurston's boundary for $T(X_0)$ introduced using the geodesic currents (cf. [23]).

Proposition 4.3. *Let X_0 be a geodesically complete infinite area hyperbolic surface with bounded geodesic pants decomposition. Then length spectrum Thurston's boundary is equal to the space of projective bounded measured laminations $PML_{bdd}(X_0)$ on X_0 .*

Proof. Consider a sequence of points $[f_k] \in T(X_0)$ that converge to (the projective class of) $L^* \in \mathbb{R}_{\geq 0}^S$ in length spectrum Thurston's boundary of $T(X_0)$. Let $t_k \rightarrow \infty$ as $k \rightarrow \infty$ be such that $\frac{1}{t_k}\mathcal{X}([f_k]) \rightarrow L^*$ in $\mathbb{R}_{\geq 0}^S - \{\bar{0}\}$, where $\bar{0}(\alpha) = 0$ for all $\alpha \in \mathcal{S}$. Let $E^{t_k \beta_k}$ be a sequence of earthquakes of \mathbb{H} such that $E^{t_k \beta_k}|_{S^1} = f_k$, where $\|\beta_k\|_{Th} < \infty$ (cf. [25]).

The proof of the above proposition gives

$$(3) \quad \left| \frac{1}{t_k} \frac{\mathcal{X}(E^{t_k \beta_k})(\alpha)}{l_{X_0}(\alpha)} - \frac{i(\beta_k, \alpha)}{l_{X_0}(\alpha)} \right| \leq \frac{1}{t_k}$$

for all $\alpha \in \mathcal{S}$.

Since $\frac{1}{t_k}\mathcal{X}([f_k]) \rightarrow L^*$, the above inequality implies

$$\left| \frac{i(\beta_k, \alpha)}{l_{X_0}(\alpha)} - \frac{L^*(\alpha)}{l_{X_0}(\alpha)} \right| \rightarrow 0$$

as $k \rightarrow \infty$ uniformly in $\alpha \in \mathcal{S}$. Define

$$\|\beta\|_{ls} = \sup_{\alpha \in \mathcal{S}} \frac{i(\beta, \alpha)}{l_{X_0}(\alpha)}$$

for any $\beta \in ML_{bdd}(X_0)$. The above convergence gives

$$\sup_{k \in \mathbb{N}} \|\beta_k\|_{ls} = N < \infty.$$

We use the assumption that X_0 has a bounded geodesic pants decomposition in order to prove that $\|\beta_k\|_{Th}$ is bounded in k . Indeed, let $\{\alpha_n\}_{n \in \mathbb{N}}$ be cuffs of a geodesic pants decomposition \mathcal{P} of X_0 such that there exists $M > 1$ with

$$\frac{1}{M} \leq l_{X_0}(\alpha_n) \leq M$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}_n$ are cuffs of \mathcal{P} . Let P^i be a geodesic pair of pants in the above decomposition with the cuffs α_{i_j} , for $j = 1, 2, 3$. Assume that α_{i_j} , for $j = 1, 2, 3$ are different geodesics of X_0 . Denote by P_{i_j} , $j = 1, 2, 3$, adjacent pair of pants to P^i with common cuff α_{i_j} . Then there exists a simple closed geodesic $\alpha_{i_j}^*$ in $P_{i_j} \cup P^i$ that intersects α_{i_j} in two points such that $l_{X_0}(\alpha_{i_j}^*)$ is bounded from the above and below by positive constants depending only on $M > 0$. The components of $P^i - \cup_{j=1}^3 (\alpha_{i_j} \cup \alpha_{i_j}^*)$ are simply connected for each i . If two of α_{i_j} , for $j = 1, 2, 3$ is the same geodesic then a similar construction yields $\alpha_{i_j}^*$ such that components of $P_i - \cup_{j=1}^3 (\alpha_{i_j} \cup \alpha_{i_j}^*)$ are simply connected and that $l_{X_0}(\alpha_{i_j}^*)$ is bounded in terms of M .

The above convergence of β_k to L^* and boundedness of the lengths of α_{i_j} and $\alpha_{i_j}^*$ on X_0 imply that

$$i(\beta_k, \alpha_{i_j}), i(\beta_k, \alpha_{i_j}^*) < C(M)$$

for some constant $C = C(M)$ and for all $i, k \in \mathbb{N}$ and $j = 1, 2, 3$. Since $X_0 - \cup_i \cup_{j=1}^3 \{\alpha_{i_j}, \alpha_{i_j}^*\}$ has simply connected and uniformly bounded components (that are polygons with at most six sides) whose boundaries are subarcs of $\alpha_{i_j}, \alpha_{i_j}^*$, we conclude that the supremum over all k and over all above components of the β_k -mass of the geodesics intersecting components is finite. Since each geodesic arc of length 1 on X_0 can intersect at most finitely many components of $X_0 - \cup_i \cup_{j=1}^3 \{\alpha_{i_j}, \alpha_{i_j}^*\}$, it follows that $\sup_{k \in \mathbb{N}} \|\beta_k\|_{Th} < \infty$.

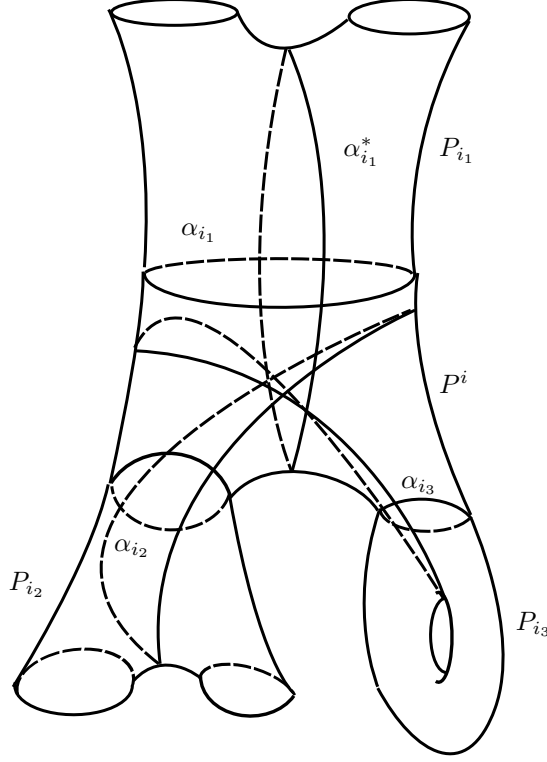
By $\sup_{k \in \mathbb{N}} \|\beta_k\|_{Th} < \infty$, there exists a subsequence β_{k_j} and $\beta^* \in ML_{bdd}(X_0)$ such that $\beta_{k_j} \rightarrow \beta^*$ as $j \rightarrow \infty$ in the weak* topology. (The weak* topology is described in terms of the lifts of the measured laminations β_k to the universal covering \mathbb{H} .) Then

$$L^*(\alpha) = \beta^*(\alpha)$$

for all $\alpha \in \mathcal{S}$ and

$$\|\beta^*\|_{Th} < \infty.$$

Thus any point in length spectrum Thurston's boundary is in $PML_{bdd}(X_0)$. The above proposition gives that all points in $PML_{bdd}(X_0)$ are also in length spectrum Thurston's boundary for $T(X_0)$. \square

FIGURE 2. Decomposition of X_0 into bounded polygons.

4.3. Infinite hyperbolic surfaces with upper bounded geodesic pants decompositions. Let X_0 be a geodesically complete infinite area hyperbolic surface with a geodesic pants decomposition $\mathcal{P} = \{\alpha_n\}_{n \in \mathbb{N}}$ such that

$$\sup_n l_{X_0}(\alpha_n) = M < \infty.$$

In addition, we assume that there exists a subsequence $\{\alpha_{n_j}\}_j$ with $l_{X_0}(\alpha_{n_j}) \rightarrow 0$ as $j \rightarrow \infty$. Let P_n^1 and P_n^2 be the geodesic pairs of pants in \mathcal{P} with a common cuff α_n (possibly $P_n^1 = P_n^2$). Let γ_n be a shortest closed geodesic in $P_n^1 \cup P_n^2$ that intersects α_n in either one point (when $P_n^1 = P_n^2$) or in two points (when $P_n^1 \neq P_n^2$). We have that (cf. [1])

$$\frac{l_{X_0}(\gamma_n)}{\max\{1, |\log l_{X_0}(\alpha_n)|\}} = O(1),$$

where $O(1)$ is a function pinched between two positive constants.

Proposition 4.4. *Let X_0 be a geodesically complete infinite area hyperbolic surface with an upper bounded geodesic pants decomposition $\mathcal{P} = \{\alpha_n\}_{n \in \mathbb{N}}$ such that a subsequence of cuffs α_{n_j} has lengths going to zero. Then length spectrum Thurston's boundary of $T(X_0)$ is strictly larger than $PML_{bdd}(X_0)$.*

Proof. We use the description of the closure of $T(X_0)$ in the Fenchel-Nielsen coordinates for the pants decomposition $\mathcal{P} = \{\alpha_n\}_{n \in \mathbb{N}}$. Namely, a marked surface f :

$X_0 \rightarrow X$ is in $T(X_0)$ if and only if its Fenchel-Nielsen coordinates $\{(\frac{l_X(\alpha_n)}{l_{X_0}(\alpha_n)}, t_X(\alpha_n))\}_{n \in \mathbb{N}}$ are uniformly bounded; $f : X_0 \rightarrow X$ is in the closure of $T(X_0)$ if and only if $\{\frac{l_X(\alpha_n)}{l_{X_0}(\alpha_n)}\}_n$ is bounded and $|t_X(\alpha_n)| = o(\max\{1, |\log l_{X_0}(\alpha_n)|\})$ for all n (cf. [20]).

Define a measured lamination $\mu = \sum_j w_j \alpha_{n_j}$ for some $w_j = o(|\log l_{X_0}(\alpha_{n_j})|)$. Then μ is not Thurston bounded and $E^{t\mu}(X_0) = X^t$ is in the closure of $T(X_0)$ for the length spectrum metric (cf. [20]). The proof of Proposition 4.2 extends to μ to get $\frac{1}{t} \mathcal{X}(X^t) \rightarrow \mu$ as $t \rightarrow \infty$ in the normalised supremum norm. Since each X^t is a limit of points in $T(X_0)$, it follows that μ is in Thurston's boundary and the proof is completed. \square

Theorem 1 from Introduction is established by Propositions 4.2, 4.3 and 4.4.

4.4. Two infinite surfaces with unbounded geodesic pants decompositions. The first surface X_1 that we consider is introduced by Kinjo [13]. Let Γ' be the hyperbolic triangle group of signature $(2, 4, 8)$. Let T' be the triangle fundamental polygon for Γ' with angles $\pi/2, \pi/4$ and $\pi/8$. Then $\Gamma'(T')$ tiles the hyperbolic plane \mathbb{H} . Let T be the union of T' and $\gamma'_0(T')$, where $\gamma'_0 \in \Gamma'$ is a reflection in the geodesic containing the side of T' which subtends the angles $\pi/2$ and $\pi/8$ of T' . Denote the vertices of T by a, b and c ; the vertex b is where T' has angle $\pi/8$ (cf. [13, Figure 2]). We choose three points a', b' and c' close to a, b and c , respectively, in the interior of the triangle T such that b' is on the side of T' containing b . The surface X_1 is obtained by puncturing the hyperbolic plane at the points $\Gamma'\{a', b', c'\}$ (cf. [13, Figures 2,3]). Kinjo [13] proved that the Teichmüller space $T(X_1)$ is complete in the length spectrum metric.

Let $\{\gamma_i\}_{i=1,\dots,8}$ be the elements of Γ' that fix a . Let l_a be the simple closed geodesic which separates the eight points $\{\gamma_i(a)\}_{i=1,\dots,8}$ from the other punctures of X_1 . We similarly define curves l_b and l_c , and then extend the definition using Γ' to all other groups of eight cusps. The lengths of all $\Gamma'(l_a)$ are the same, as well as the lengths of all $\Gamma'(l_b)$, as well as the lengths of all $\Gamma'(l_c)$.

For the triangle T , we denote by $l_{a',b'}$ the simple closed geodesic which is homotopic to a simple closed curve in T that separates a', b' from c' . We similarly extend the definition to $l_{b',c'}$ and $l_{c',a'}$, and then extend it to all triangles using the invariance under Γ' . Note that the lengths of $\Gamma'(l_{a',b'})$ are the same, as well as the lengths of all $\Gamma'(l_{b',c'})$, and the lengths of all $\Gamma'(l_{c',a'})$.

The lengths of the family of geodesics $\Gamma'(l_a) \cup \Gamma'(l_b) \cup \Gamma'(l_c) \cup \Gamma'(l_{a',b'}) \cup \Gamma'(l_{b',c'}) \cup \Gamma'(l_{c',a'})$ are bounded from the below and from the above, and this family separates the surface X_1 into finite bounded polygons with uniformly bounded number of sides. Then the proof of Proposition 4.4 extends to show that length spectrum Thurston's boundary coincides with $PML_{bdd}(X_1)$.

Denote by X_2 an infinite hyperbolic surface defined by Shiga [24] that has geodesic pants decomposition with cuff lengths converging to infinity. The surface X_2 contains a sequence γ_n of simple closed geodesics with $l_{X_2}(\gamma_n) \rightarrow \infty$ as $n \rightarrow \infty$ such that for each closed geodesic δ we have

$$(4) \quad l_{X_2}(\delta) \geq \sum_{k=1}^{\infty} k l_{X_2}(\gamma_k) i(\gamma_k, \delta),$$

where only finitely many terms are non-zero. Shiga [24] proved that a sequence of full Dehn twists f_n around the curve γ_n diverges in the Teichmüller metric and

it converges to the identity in the length spectrum metric. Thus the two metrics produce different topologies on $T(X_2)$.

We define β_n to be a measured lamination whose support is $\{\gamma_k\}_{k=1,\dots,n}$ such that, for $k = 1, \dots, n$,

$$\beta_n|_{\gamma_k} = l_{X_2}(\gamma_k).$$

The projective class $[\beta_n]$ is in $PML_{bdd}(X_2)$. Define β_* to be a measured lamination on X_2 whose support is $\{\gamma_k\}_{k=1}^\infty$ such that, for all $k = 1, 2, \dots$,

$$\beta_*|_{\gamma_k} = l_{X_2}(\gamma_k).$$

It is clear that the projective class $[\beta_*]$ is not in $PML_{bdd}(X_2)$.

We prove that $[\beta_n] \rightarrow [\beta_*]$ as $n \rightarrow \infty$ in the normalized supremum norm. Indeed, let δ be a simple closed geodesic in X_2 . Then

$$\frac{|i(\beta_n, \delta) - i(\beta_*, \delta)|}{l_{X_2}(\delta)} = \sum_{k=n+1}^{\infty} \frac{i(\beta_k, \delta)}{l_{X_2}(\delta)} = \frac{\sum_{k=n+1}^{\infty} i(\delta, \gamma_k) l_{X_2}(\gamma_k)}{\sum_{k=1}^{\infty} k i(\delta, \gamma_k) l_{X_2}(\gamma_k)} \leq \frac{1}{n+1}$$

and $[\beta_*]$ is in length spectrum Thurston's boundary of $T(X_2)$. Therefore the boundary is larger than $PML_{bdd}(X_2)$.

Open problem: Assume that a sequence in $T(X_0)$ converges to a bounded projective measured lamination in length spectrum Thurston's boundary. Is it true that the sequence converges in Thurston's boundary introduced using geodesic currents?

REFERENCES

- [1] D. Alessandrini, L. Liu, A. Papadopoulos, W. Su and Z. Sun, *On Fenchel-Nielsen coordinates on Teichmüller spaces of surfaces of infinite type*, Ann. Acad. Sci. Fenn. Math. 36 (2011), no. 2, 621-659.
- [2] D. Alessandrini, L. Liu, A. Papadopoulos and W. Su, *On the inclusion of the quasiconformal Teichmüller space into the length-spectrum Teichmüller space*, preprint, arXiv:1201.6030.
- [3] A. Basmajian and Y. Kim, *Geometrically infinite surfaces with discrete length spectra*, Geom. Dedicata 137 (2008), 219-240.
- [4] A. Basmajian and D. Šarić, *Geodesically Complete Hyperbolic Structures*, preprint.
- [5] A. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, 91. Springer-Verlag, New York, 1983.
- [6] F. Bonahon, *The geometry of Teichmüller space via geodesic currents*, Invent. Math. 92 (1988), no. 1, 139-162.
- [7] P. Buser, *Geometry and spectra of compact Riemann surfaces*, Reprint of the 1992 edition. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2010.
- [8] D. B. A. Epstein and A. Marden, *Convex hulls in hyperbolic space, a theorem of Sullivan and measured pleated surfaces*, LMS Lecture Notes 111, pages 112-253, Cambridge University Press, 1987.
- [9] D. B. A. Epstein, A. Marden and V. Markovic, *Quasiconformal homeomorphisms and the convex hull boundary*, Ann. of Math. (2) 159 (2004), no. 1, 305-336.
- [10] A. Fathi, F. Laudenbach and V. Poénaru, *Thurston's work on surfaces*, Translated from the 1979 French original by Djun M. Kim and Dan Margalit. Mathematical Notes, 48. Princeton University Press, Princeton, NJ, 2012.
- [11] F. Gardiner and N. Lakic, *Quasiconformal Teichmüller Theory*, Mathematical Surveys and Monographs, Volume 76, A.M.S. 2000.
- [12] F. Gardiner, J. Hu and N. Lakic, *Earthquake curves*, Complex manifolds and hyperbolic geometry (Guanajuato, 2001), 141-195, Contemp. Math., 311, Amer. Math. Soc., Providence, RI, 2002.

- [13] E. Kinjo, *On Teichmüller metric and the length spectrums of topologically infinite Riemann surfaces*, Kodai Math. J. 34 (2011), no. 2, 179-190.
- [14] R. Mañé, P. Sad and D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. Ecole Norm. Sup, 16, 193-217, 1983.
- [15] K. Matsuzaki, *A classification of the modular transformations of infinite dimensional Teichmüller spaces*, In the tradition of Ahlfors-Bers. IV, 167-177, Contemp. Math., 432, Amer. Math. Soc., Providence, RI, 2007.
- [16] H. Miyachi and D. Šarić, *Uniform weak* topology and earthquakes in the hyperbolic plane*, Proc. Lond. Math. Soc. (3) 105 (2012), no. 6, 1123-1148.
- [17] J. P. Otal, *About the embedding of Teichmüller space in the space of geodesic Hölder distributions*, Handbook of Teichmüller theory. Vol. I, 223-248, IRMA Lect. Math. Theor. Phys., 11, Eur. Math. Soc., Zürich, 2007.
- [18] D. Šarić, *Real and Complex Earthquakes*, Trans. Amer. Math. Soc. 358 (2006), no. 1, 233-249.
- [19] D. Šarić, *Bounded earthquakes*, Proc. Amer. Math. Soc. 136 (2008), no. 3, 889-897.
- [20] D. Šarić, *Fenchel-Nielsen coordinates on upper bounded pants decompositions*, to appear Math. Proc. Cambridge Philos. Soc., arXiv:1209.5819.
- [21] D. Šarić, *Geodesic currents and Teichmüller spaces*, Topology 44 (2005), no. 1, 99-130.
- [22] D. Šarić, *Earthquakes in the length spectrum Teichmüller space*, Proc. Amer. Math. Soc. 143 (2015), no. 4, 1531-1543.
- [23] D. Šarić, *Thurston's boundary for Teichmüller spaces of infinite surfaces: geodesic currents*, preprint, available on arxiv.org.
- [24] H. Shiga, *On a distance defined by the length spectrum of Teichmüller space*, Ann. Acad. Sci. Fenn. Math. 28 (2003), no. 2, 315-326.
- [25] W. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.) 19 (1988), no. 2, 417-431.
- [26] W. Thurston, *Earthquakes in two-dimensional hyperbolic geometry*, Low-dimensional topology and Kleinian groups (Coventry/Durham, 1984), 91-112, London Math. Soc. Lecture Note Ser., 112, Cambridge Univ. Press, Cambridge, 1986.
- [27] S. Wolpert, *The Fenchel-Nielsen deformation*, Ann. of Math. (2) 115 (1982), no. 3, 501-528.

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